Controllability of a semi discretized parabolic equation via the moment method.

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WHAT WE ARE INTERESTED IN

Discrete control theory on a semi-discretized parabolic equation on $\Omega = (0,1)$.
$\mathcal{A}^h$: discretization of an elliptic operator (example : $\mathcal{A} = -\Delta$).

$$\begin{cases} \partial_t y^h(t) + \mathcal{A}^h y^h(t) = V_{\mathrm{d}}^h(t)\mathbf{1}_\omega, \text{ on } (0,T), \ (\omega \subset \Omega), \\ y^h(0) = y^{h,0} \in \mathbb{R}^N, \\ y_0^h(t) = 0, \text{ on } (0,T), \\ y_{N+1}^h(t) = V_{\mathrm{b}}^h(t), \text{ on } (0,T), \end{cases}$$

Find $V_{\mathrm{d}}^h \in L^2(0,T;\mathbb{R}^N)$ OR $V_{\mathrm{b}}^h \in L^2(0,T;\mathbb{R})$:

- $y^h(T) = 0$
- V_{d}^h, V_{b}^h uniformly bounded w.r.t. h.

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WHAT WAS DONE BEFORE

1998, López and Zuazua

- semi-discretized heat equation : $\mathcal{A}^h = -\Delta^h$,
- uniform mesh,
- boundary null-control problem : V_{b}^h,
- in space dimension 1 : $\Omega = (0,1)$.

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WHAT WAS DONE BEFORE

2010, Boyer, Hubert and Le Rousseau

- semi-discretized parabolic equation : $\mathcal{A}^h = (-\partial_x(\gamma\partial_x \cdot))^h$,
- distributed control problem : V_{d}^h, (relaxed control)
- in space dimension ≥ 1,
- discrete Carleman estimates.

WHAT WE DO

Extend their work to :

- Cascade system of parabolic equations : $\begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix}$ with $\begin{pmatrix} \text{control} \\ 0 \end{pmatrix}$
- boundary and distributed controls : V_{d}^h, V_{b}^h,
- BUT : in space dimension 1.

Outline.

1. The moment method on a semi-discretized parabolic equation.
2. Discrete spectral properties.
3. Application in control theory

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DISCRETE PROBLEM

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(\mathrm{P}^h)\begin{cases} (y^h)'(t) + \mathcal{A}^h y^h(t) = 0, \text{ on } (0,T), \\ y^h(0) = y^{h,0} \in \mathbb{R}^N \\ y_0^h(t) = 0, \text{ on } (0,T), \\ y_{N+1}^h(t) = 0, \text{ on } (0,T). \end{cases}
$$

ELLIPTIC OPERATOR

- $\mathcal{A}^h := \left(-\frac{\partial}{\partial x}\left(\gamma\frac{\partial}{\partial x}\cdot\right) + q\right)^h,$
- $(\mathcal{A}^h y^h)_j = -\frac{1}{h}\left(\gamma_{j+1/2}\frac{y_{j+1}^h(t)-y_j^h(t)}{h} - \gamma_{j-1/2}\frac{y_j^h(t)-y_{j-1}^h(t)}{h}\right) + q_j y_j^h(t)$
- Denote by $(\Lambda^h := (\lambda_k^h)_{k=1}^N, (\phi_k^h)_{k=1}^N)$ the eigenelements of $\mathcal{A}^h$, $\|\phi_k^h\|_h = 1$.

PARAMETERS

- $q \in C^0(\Omega)$,
- $\gamma \in C^2(\Omega)$, $\gamma \geq \gamma_{min} > 0$,
- $V_\mathrm{b} \in L^2(0,T;\mathbb{R})$.

The null-control problem.

DISCRETE PROBLEM

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DISCRETE PROBLEM

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(\mathrm{P}^h)\begin{cases}
(y^h)'(t) + \mathcal{A}^h y^h(t) = 0 + \gamma_{N+1/2}\dfrac{V_b^h(t)}{h^2}\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}, \text{ on } (0,T),\\
y^h(0) = y^{h,0} \in \mathbb{R}^N\\
y_0^h(t) = 0, \text{ on } (0,T),\\
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\end{cases}
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DISCRETE PROBLEM

$$(\mathrm{P}^h)\begin{cases}(y^h)'(t)+\mathcal{A}^h y^h(t)=0+\gamma_{N+1/2}\dfrac{V_b^h(t)}{h^2}\mathbf{e_N}, \text{ on } (0,T),\\ y^h(0)=y^{h,0}\in\mathbb{R}^N\\ y_0^h(t)=0, \text{ on } (0,T),\\ y_{N+1}^h(t)=V_\mathrm{b}(t)\in L^2(0,T;\mathbb{R}), \text{ on } (0,T).\end{cases}$$

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PARAMETERS

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PROPERTY OF THE SOLUTION

- $\displaystyle\int_0^T \left(e^{-\lambda_k^h (T-t)} \phi_k^h, \left[(y^h)'(t) + \mathcal{A}^h y^h(t) = 0 + \gamma_{N+1/2} \frac{V_b^h(t)}{h^2} \mathbf{e_N} \right] \right) \mathrm{d}t,$
- Integrate by parts,

$$\left(y^h(T), \phi_k^h \right) - \left(y_0^h, e^{-\lambda_k^h T} \phi_k^h \right) = -\gamma_{N+1/2} \left(\frac{0 - (\phi_k^h)_N}{h} \right) \int_0^T e^{-\lambda_k^h (T-t)} V_b^h(t) \mathrm{d}t$$

$$\boxed{\begin{gathered} y^h(T) = 0 \\ \Updownarrow \\ \forall k \in \{1, \ldots N\}, \; -\left(y_0^h, e^{-\lambda_k^h T} \phi_k^h \right) = -\gamma_{N+1/2} \left(\frac{0 - (\phi_k^h)_N}{h} \right) \int_0^T e^{-\lambda_k^h (T-t)} V_b^h(t) \mathrm{d}t \end{gathered}}$$

MOMENT PROBLEM

Find V_d^h and V_b^h, uniformly bounded in h, such that :

$$\forall k \in \{1, \ldots N\}, \; -\left(y_0^h, e^{-\lambda_k^h T} \phi_k^h \right) = \begin{cases} -\gamma_{N+1/2} \dfrac{0 - (\phi_k^h)_N}{h} \displaystyle\int_0^T e^{-\lambda_k^h (T-t)} \overbrace{V_b^h(t)}^{\in \mathbb{R}} \mathrm{d}t \\ \displaystyle\int_0^T e^{-\lambda_k^h (T-t)} (\underbrace{V_d^h(t)}_{\in \mathbb{R}^N}, \mathbf{1}_\omega \phi_k^h) \mathrm{d}t \end{cases}$$

Definition : Biorthogonal family

Let $\Sigma := (\sigma_k)_{k\geq 1}$ be a sequence of positive real numbers.
Biorthogonal family for Σ, $(q_j^\Sigma)_{j\geq 1}$:

$$\forall k, j \geq 1,\ q_j^\Sigma \in L^2(0,T), \quad \int_0^T e^{-\sigma_k(T-t)} q_j^\Sigma(t)\mathrm{d}t = \delta_{k,j}.$$

Recall the problem : $(\Lambda^h := (\lambda_k^h)_{k\geq 1}$: eigenvalues of $\mathcal{A}^h)$

$$-\left(y_0^h, e^{-\lambda_k^h T}\phi_k^h\right) = \begin{cases} \displaystyle\int_0^T e^{-\lambda_k^h(T-t)}\left(V_{\mathrm{d}}^h(t), \mathbf{1}_\omega \phi_k^h\right)\mathrm{d}t \\ \displaystyle -\gamma_{N+1/2}\left(\frac{0-(\phi_k^h)_N}{h}\right)\int_0^T e^{-\lambda_k^h(T-t)} V_{\mathrm{b}}^h(t)\mathrm{d}t \end{cases}$$

POSSIBLE EXPRESSIONS FOR THE CONTROLS

$$V_{\mathrm{d}}^h(t) = \sum_{j=1}^N \frac{-\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)}{\|\mathbf{1}_\omega \phi_j^h\|_h^2}\phi_j^h q_j^{\Lambda^h}(t),$$

$$V_{\mathrm{b}}^h(t) = \sum_{j=1}^N \frac{\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)}{\gamma_{N+1/2}\left(\frac{0-\phi_{j,N}^h}{h}\right)} q_j^{\Lambda^h}(t).$$

The moment method, part 3/3 : justifications

IT REMAINS TO PROVE

- Uniform bounds on V_b^h and V_d^h $\Leftarrow$ $\boxed{\|\mathbf{1}_\omega \phi_j^h\|_h^2 \geq \ ? \text{ and } \left(\frac{0-\phi_{j,N}^h}{h}\right) \geq \ ?}$
- Bounds/existence of $(q_j^{\Lambda^h})_{j\geq 1}$ for all $h > 0$

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IT REMAINS TO PROVE

- Uniform bounds on V_{b}^h and V_{d}^h $\Leftarrow$ $\boxed{\|\mathbf{1}_\omega \phi_j^h\|_h^2 \geq \text{ ? and } \left(\frac{0-\phi_{j,N}^h}{h}\right) \geq \text{ ?}}$
- Bounds/existence of $(q_j^{\Lambda^h})_{j\geq 1}$ for all $h>0$ $\Leftarrow$ $\boxed{\text{find } \rho,\, \mathcal{N} : \forall h>0,\, \Lambda^h \in \mathcal{L}(\rho,\mathcal{N})}$

Definition : set of sequences $\mathcal{L}(\rho,\mathcal{N})$

Let $\rho>0$ and $\mathcal{N}:\mathbb{R}^+\to\mathbb{N}$.
Denote by $\mathcal{L}(\rho,\mathcal{N})$ the set of sequences $\Sigma=(\sigma_k)_{k\geq 1}$ such that :

- $\forall k\geq 1,\ \sigma_{k+1}-\sigma_k \geq \rho,$
- $\forall \varepsilon>0,\ \displaystyle\sum_{k=\mathcal{N}(\varepsilon)}^{\infty} \frac{1}{\sigma_k} \leq \varepsilon.$

Theorem [Fattorini-Russel, 1974]

Let $\rho>0$ and $\mathcal{N}:\mathbb{R}^+\to\mathbb{N}$.

$$\forall\varepsilon>0,\ \exists K_\varepsilon>0,\ \boxed{\forall\Sigma\in\mathcal{L}(\rho,\mathcal{N})},\ \exists(q_k^\Sigma)_{k\geq 1},\ \forall k\geq 1,\ \|q_k^\Sigma\|_{L^2}\leq K_\varepsilon \exp(\varepsilon\sigma_k).$$

where (q_k^Σ) is a biorthogonal family for Σ.

WHEN $\gamma = 1$ AND $q = 0$: $\mathcal{A} = -\Delta$ (UNIFORM MESH)

Theorem [López-Zuazua,1998], boundary control problem.

The moment method on the semi-discretized heat equation gives uniformly bounded control :

$$\|V_{\mathrm{b}}^h\|_{L^2(0,T;\mathbb{R})} \leq C_T \|y^{h,0}\|.$$

for the null-control problem (P^h).

PROOF Explicit expression for the eigenelements !

$$\forall k \in \{1, \dots, N\},\ \lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{\pi h k}{2}\right)$$

Extend the sequence :

$$(\lambda_k^h)_{k\geq 1} = \begin{cases} \dfrac{4}{h^2}\sin^2(\dfrac{\pi h k}{2}), \text{ for } k \in \{1, \dots N\}, & \text{(discrete eigenvalues)} \\ k^2\pi^2, \text{ for } k \geq N+1. & \text{(continuous eigenvalues)} \end{cases}$$

There exist $\rho > 0$, and $\mathcal{N}$ such that

$$\boxed{\forall h > 0,}\ \Lambda^h := (\lambda_k^h)_{k\geq 1} \in \mathcal{L}(\rho, \mathcal{N}).$$

and since : $\phi_k^h = (\sin(j\pi h k))_{j=1}^N$, we can estimate $\left|\frac{0-(\phi_k^h)_N}{h}\right| \geq \frac{2}{\pi}\sqrt{\lambda_k^h}$.

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γ AND q IN THE GENERAL CASE ?

Can one obtain the same results with a general operator $\mathcal{A} = -\frac{\partial}{\partial x}\left(\gamma \frac{\partial}{\partial x}\cdot\right) + q\cdot$?
No explicit formulae for the eigenelements.

STRATEGY

- Find ρ and $\mathcal{N}$ such that : $\forall h > 0$, $\Lambda^h \in \mathcal{L}(\rho, \mathcal{N})$.
- Find lower bounds on $\left|\frac{0-(\phi_k^h)_N}{h}\right|$ and $\|\mathbf{1}_\omega \phi_k^h\|$.

Outline.

PROBLEME

Find sharp lower-bounds for : $\left|\frac{0-(\phi_k^h)_N}{h}\right|$ and $\|\mathbf{1}_\omega \phi_k^h\|$.

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i,j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \tag{1}$$

then the following relations holds : $\left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$ and $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$.

Prove (2) with a sharp constant C_k.

PROOF How to prove (2) ?

CONTINUOUS SETTING

EINGENVALUE PROBLEME FOR $\mathcal{A} := -\partial_x(\gamma\partial_x\cdot) + q\cdot$

- ODE of ordre 2 : $\mathcal{A}\phi_k = \lambda_k\phi_k \longrightarrow$ system of ODEs of dimension 2.
- **CHANGE OF VARIABLE** : $\Phi_k(x) = \begin{pmatrix} \phi_k(x) \\ \gamma(x)\phi_k'(x) \end{pmatrix}$
- We get the relation : $\Phi_k'(x) = \begin{pmatrix} 0 & 1/\gamma(x) \\ -\lambda_k & 0 \end{pmatrix} \Phi_k(x) + \begin{pmatrix} 0 & 0 \\ q(x) & 0 \end{pmatrix} \Phi_k(x)$
- Set $S(x, x_0) = \exp\left(\int_{x_0}^{x} \begin{pmatrix} 0 & 1/\gamma(x) \\ -\lambda_k & 0 \end{pmatrix}\right)$.
- Duhamel formula :
  $\Phi_k(x) = S(x, x_0)\Phi_k(x_0) + \displaystyle\int_{x_0}^{x} S(x, s) \begin{pmatrix} 0 & 0 \\ q(x) & r(x) \end{pmatrix} \Phi_k(s)\mathrm{d}s$.
- Gronwall's lemma : $\|\Phi_k(x)\| \leq \|S(x, x_0)\Phi_k(x_0)\| \exp\left(\int_{x_0}^{x} \|q\|_\infty \|S(x, s)\|\mathrm{d}s\right)$.

$$\|S(x, x_0)\| \leq ?$$

PROOF How to prove (2) ?

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- **CHANGE OF VARIABLE** : $\Phi_k(x) = \begin{pmatrix} \phi_k(x) \\ \sqrt{\frac{\gamma(x)}{\lambda_k}}\phi_k'(x) \end{pmatrix}$
- We get the relation :

$$\Phi_k'(x) = \begin{pmatrix} 0 & \sqrt{\frac{\lambda_k}{\gamma(x)}} \\ -\sqrt{\frac{\lambda_k}{\gamma(x)}} & 0 \end{pmatrix} \Phi_k(x) + \begin{pmatrix} 0 & 0 \\ q(x) & \underbrace{\frac{1}{\sqrt{\gamma(x)}}\left(\frac{1}{\sqrt{\gamma}}\right)'(x)}_{:=r(x)} \end{pmatrix} \Phi_k(x)$$

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- Gronwall's lemma : $\|\Phi_k(x)\| \leq \|S(x,x_0)\Phi_k(x_0)\| \exp\left(\int_{x_0}^{x} C\|S(x,s)\|\mathrm{d}s\right)$.

$$\boxed{\|S(x,x_0)\| = 1 \Rightarrow |\phi(x)| + \frac{1}{\sqrt{\lambda_k}}\left|\phi'(x)\right| \leq e^C\left(|\phi(x_0)| + \frac{1}{\sqrt{\lambda_k}}\left|\phi'(x_0)\right|\right)}$$

PROOF How to prove (2) ?

DISCRETE SETTING

EINGENVALUE PROBLEME FOR $\mathcal{A}^h$

- "ODE" of ordre 2 : $\mathcal{A}^h \phi_k^h = \lambda_k^h \phi_k^h \longrightarrow$ system of "ODEs" of dimension 2.
- **CHANGE OF VARIABLE** $(\Phi_k^h)_j = \begin{pmatrix} (\phi_k^h)_j \\ \frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h} \frac{\sqrt{\gamma_{j-1/2}}}{\sqrt{\lambda_k^h}} \end{pmatrix}$
- Duhamel's formula + Gronwall's lemma :
  $\forall 1 \leq i,j \leq N,\ \|(\Phi_k^h)_j\| \leq C \max_{1\leq i,j\leq N} \|S_{i,j}^k\| \|(\Phi_k^h)_i\|.$

  where : $S_{i,j}^k = (I_h + hM_{\lambda_k^h, i-1})(I_h + hM_{\lambda_k^h, i-2}) \dots (I_h + hM_{\lambda_k^h, j}),$

  and : $M_{\lambda_k^h, j} := \begin{pmatrix} -h\frac{\lambda_k^h}{\gamma_{j+1/2}} & \sqrt{\frac{\lambda_k^h}{\gamma_{j+1/2}}} \\ -\sqrt{\frac{\lambda_k^h}{\gamma_{j+1/2}}} & 0 \end{pmatrix}$
- Thus,

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq \max_{1\leq i,j\leq N} \|S_{i,j}^k\| \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)}$$

Discrete bounds on eigenvectors.

Proposition : Estimates on $S_{i,j}^k$

Estimates on the semi-group $S_{i,j}^k$ for all i, j :

- **For any** k **:** $\quad \|S_{i,j}^k\| \leq e^{C\sqrt{\lambda_k^h}}$,

Proposition : Estimates on the eigenvectors

- **For any** k **:** $\left|\frac{(\phi_k^h)_N}{h}\right| \geq C_1 e^{-C_2\sqrt{\lambda_k^h}}$ and $h\sum_{jh\in\omega} |(\phi_k^h)_j|^2 \geq C_1 e^{-C_2\sqrt{\lambda_k^h}}$

Proposition : Gap property

- **For any** k **:** NO UNIFORM GAP PROPERTY.

Discrete bounds on eigenvectors.

Proposition : Estimates on $S^k_{i,j}$

Estimates on the semi-group $S^k_{i,j}$ for all i, j :

- **For any k :** $\|S^k_{i,j}\| \leq e^{C\sqrt{\lambda^h_k}}$,

Define

$$k^h_{\max} := \max\left\{k \in \{1, \ldots N\};\ \lambda^h_k < \frac{4}{h^2}\gamma_{min}(1-\varepsilon)\right\}.$$

- **For $k \leq k^h_{\max}$:** $\|S^k_{i,j}\| \leq \frac{1}{\delta_\varepsilon}$

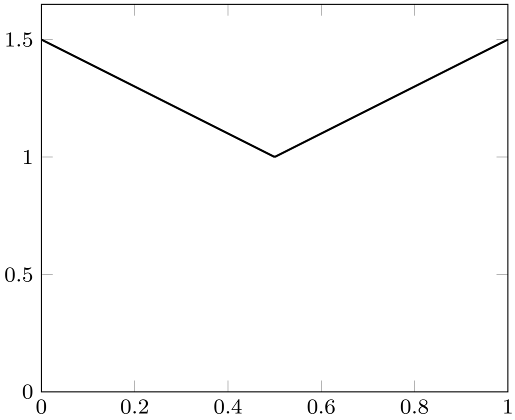
Proposition : Estimates on the eigenvectors

- **For any k :** $\left|\frac{(\phi^h_k)_N}{h}\right| \geq C_1 e^{-C_2\sqrt{\lambda^h_k}}$ and $h\sum_{jh\in\omega} |(\phi^h_k)_j|^2 \geq C_1 e^{-C_2\sqrt{\lambda^h_k}}$
- **For $k \leq k^h_{\max}$:** $\left|\frac{(\phi^h_k)_N}{h}\right| \geq \delta_\varepsilon\sqrt{\lambda^h_k}$ and $h\sum_{jh\in\omega} |(\phi^h_k)_j|^2 \geq \delta_\varepsilon$

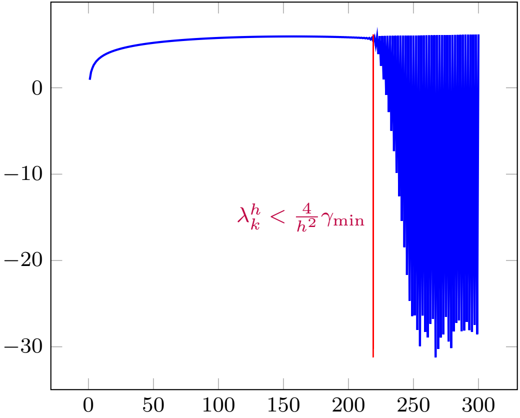
Proposition : Gap property

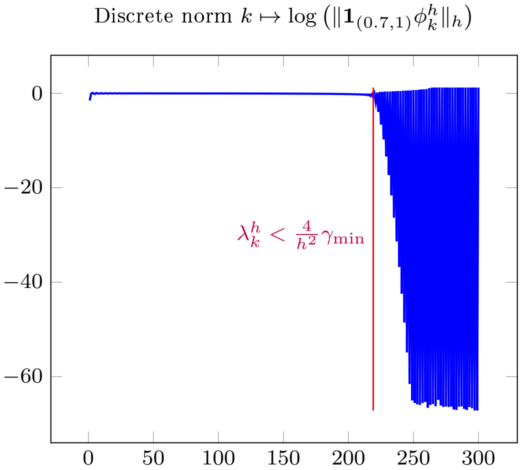
- **For any k :** NO UNIFORM GAP PROPERTY.
- **For $k \leq k^h_{\max}$:** $\lambda^h_{k+1} - \lambda^h_k \geq \delta_\varepsilon$

Function $x \mapsto \gamma(x)$

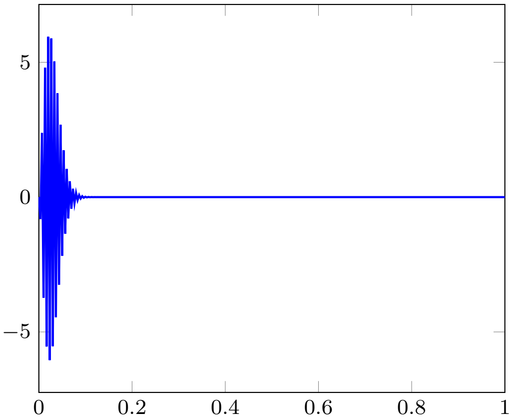


Discrete derivative: $k \mapsto \log\left(\left|\frac{0-(\phi_k^h)_N}{h}\right|\right)$

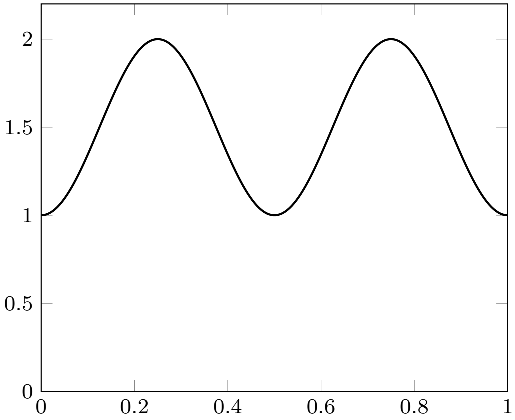


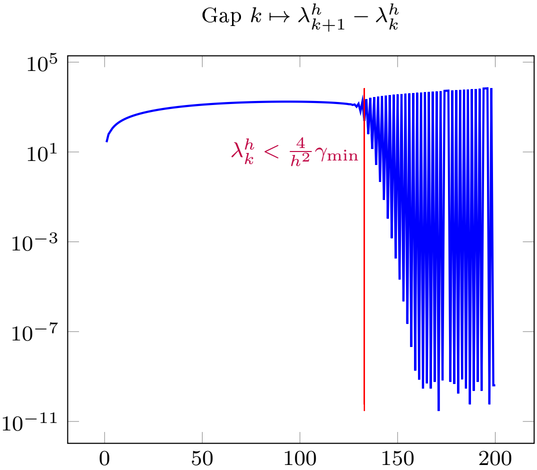

Discrete norm $k \mapsto \log\left(\|\mathbf{1}_{(0.7,1)}\phi_k^h\|_h\right)$
0
−20
−40
−60
0
50
100
150
200
250
300
$\lambda_k^h < \frac{4}{h^2}\gamma_{\min}$

Shape of the $(N-1)^{\text{th}}$ eigenfunction



Function $x \mapsto \gamma(x)$




Gap $k \mapsto \lambda_{k+1}^h - \lambda_k^h$
10^5
10^1
10^{-3}
10^{-7}
10^{-11}
0
50
100
150
200
$\lambda_k^h < \frac{4}{h^2}\gamma_{\min}$

Outline.

Recap.

EXPRESSIONS OF THE CONTROLS

$$V_{\mathrm{d}}^h(t) = \sum_{j=1}^{N} \frac{-\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)_h}{\|\mathbf{1}_\omega \phi_j^h\|_h^2} \phi_j^h q_j^{\Lambda^h}(t),$$

$$V_{\mathrm{b}}^h(t) = \sum_{j=1}^{N} \frac{\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)_h}{\gamma_{N+1/2}\left(\frac{0-\phi_{j,N}^h}{h}\right)} q_j^{\Lambda^h}(t).$$

RECALL THE STRATEGY

- Find lower bounds on $\left|\frac{0-(\phi_k^h)_N}{h}\right|$ or $\|\mathbf{1}_\omega \phi_k^h\|$: **OK.**
- Find ρ and $\mathcal{N}$ such that : $\forall h > 0$, $\Lambda^h \in \mathcal{L}(\rho, \mathcal{N})$: KO.

TO SUM UP

- **For all k,** $h\sum_{jh\in\omega} |(\phi_k^h)_j|^2 \geq C_1 e^{-C_2\sqrt{\lambda_k^h}}$.
- **For all k,** $\left|\frac{(\phi_k^h)_N}{h}\right| \geq C_1 e^{-C_2\sqrt{\lambda_k^h}}$.
- If $k \leq k_{\max}^h$, then $\lambda_{k+1}^h - \lambda_k^h \geq \delta_\varepsilon$.

Partial controlability result.

Theorem [A.-Boyer-Morancey, 2016]

We say that relaxed control up to rank $k^h_{\max}$ holds for system (P^h) if :
$\forall T > 0$, there exists a control V^h_{d} (or V^h_{b}) satisfying

$$\forall h > 0,\ \|V^h_{\mathrm{d}}\| \leq C\|y^{h,0}\| \quad (\text{or } \|V^h_{\mathrm{b}}\| \leq C\|y^{h,0}\|)$$

and such that the corresponding solution verifies :

$$\forall h > 0,\ \|y^h(T)\| \leq C\|y^{h,0}\|e^{-\frac{T}{2}\lambda^h_{k^h_{\max}}}.$$

Let $\varepsilon > 0$ and let $k^h_{\max}$ be such that $\lambda^h_{k^h_{\max}} < \frac{4}{h^2}\gamma_{min}(1-\varepsilon)$.
Relaxed controllability up to rank $k^h_{\max}$ holds for system (P^h).

Remarks

- *The solution satisfies in fact :* $\forall h > 0,\ \|y^h(T)\| \leq \|y^{h,0}\|C_1 e^{-\frac{C_2}{h^2}}$.
- *Simpler proof of known results with a wider range of applications.*

System of two parabolic equations in one space dimension, $\Omega = (0, L)$.
Only one control force on the first equation (distributed or boundary).

$$(\mathrm{S}^h)\begin{cases} (y^h)'(t) + \begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix} y^h(t) = \begin{pmatrix} V_{\mathrm{d}}^h \mathbf{1}_\omega \\ 0 \end{pmatrix} + \gamma_{N+1/2} \dfrac{V_b^h(t)}{h^2} (\mathbf{e_N} 0), \text{ on } (0,T), \\ y^h(0) = y^{h,0} \in \mathbb{R}^N \\ y_0^h(t) = 0, \text{ on } (0,T), \end{cases}$$

Note that the second equation is controlled by the solution to the first one.

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Theorem [A.-Boyer-Morancey, 2016]

Let $\varepsilon > 0$ and let $k^h_{\max}$ be such that $\lambda^h_{k^h_{\max}} < \frac{4}{h^2}\gamma_{min}(1-\varepsilon)$.

Relax controllability up to rank $k^h_{\max}$ holds for system (S^h).

Remarks

The Carleman technics employed by [2010, Boyer, Hubert and Le Rousseau] cannot be used here.

System of two parabolic equations in one space dimension, $\Omega = (0, L)$.
Only one control force on the first equation (distributed or boundary).

$$(\mathrm{S}^h) \begin{cases} (y^h)'(t) + \begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix} y^h(t) = \begin{pmatrix} V_{\mathrm{d}}^h \mathbf{1}_\omega \\ 0 \end{pmatrix} + \gamma_{N+1/2} \dfrac{V_b^h(t)}{h^2} \left(\mathbf{e_N} 0\right), \text{ on } (0,T), \\ y^h(0) = y^{h,0} \in \mathbb{R}^N \\ y_0^h(t) = 0, \text{ on } (0,T), \end{cases}$$

Elements of proof.

Main difference with the scalar case :

- Operator $\begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix}$ is not diagonalizable $\Rightarrow$ we use the Jordan form.
- Existence + estimates of a biorthogonal family for $\left(e^{-\lambda_k^h t}\right)_{k\geq 1} \cup \left(te^{-\lambda_k^h t}\right)_{k\geq 1}$.

$\square$

SUM UP

We have built an elementary approach :

- to solve the control problem for a large class of parabolic equations,
- which applies on quasi-uniform meshes,
- $\boxed{\text{which applies on a parabolic cascade system,}}$
  (with fewer controls than equations)
- only valid in 1D.

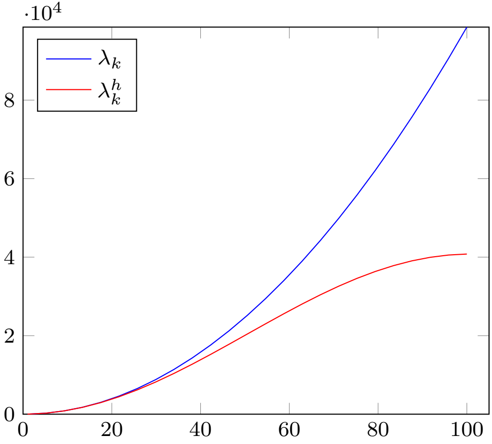
PERSPECTIVE

Cascade systems with variable coefficients.

Thank you for your attention!

Basic approach : One could have tried to use numerical analysis $\lambda_k^h \approx \lambda_k$.

$\boxed{\lambda_k^h \approx \lambda_k \Longrightarrow \text{ Gap property only for a portion of the spectrum.}}$



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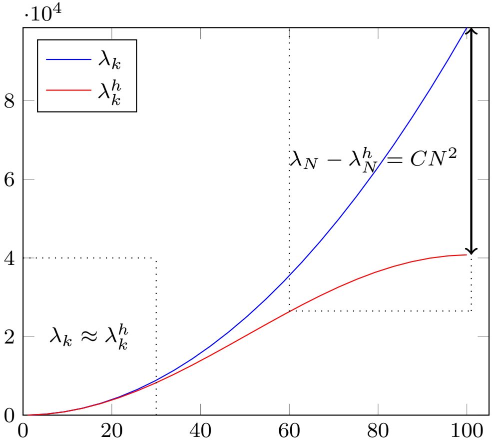

Definition : set of sequences $\mathcal{L}(\rho,\mathcal{N})$

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.
Denote by $\mathcal{L}(\rho,\mathcal{N})$ the set of sequences $\Sigma = (\sigma_k)_{k\geq 1}$ such that :

- $\forall k \geq 1$, $\sigma_{k+1} - \sigma_k \geq \rho$,
- $\forall \varepsilon > 0$, $\displaystyle\sum_{k=\mathcal{N}(\varepsilon)}^{\infty} \frac{1}{\sigma_k} \leq \varepsilon$.

Theorem [Fattorini-Russel, 1974]

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.

$$\forall \varepsilon > 0,\ \exists K_\varepsilon > 0,\ \boxed{\forall \Sigma \in \mathcal{L}(\rho,\mathcal{N})},\ \exists (q_k^\Sigma)_{k\geq 1},\ \forall k \geq 1,\ \|q_k^\Sigma\|_{L^2} \leq K_\varepsilon \exp(\varepsilon\sigma_k).$$

where (q_k^Σ) is a biorthogonal family for Σ.

[Ammar Khodja - Benabdallah - González Burgos - de Teresa, 2011]
Let $m \in \mathbb{N}$, we have the same results for the family $(t^j e^{-\sigma_k t})_{m\geq j\geq 0, k\geq 1}$.

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i, j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \qquad (2)$$

then the following relations holds : $\left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$ and $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$.

PROOF (SKETCH)

$$\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)$$

$$\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left|(\phi_k^h)_j\right| \quad \text{now : } h\sum_{j=1}^{N} \cdot$$

$$\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k. \text{ Take } i = N+1 : \left|\frac{0 - (\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$$

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i, j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \qquad (2)$$

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PROOF (SKETCH) Now : $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$?

CONTINOUS CASE : STEP 1

Find a nodal domain (a, b) in ω : $\phi_k(a) = \phi_k(b) = 0$

$$\int_a^b -\partial_x(\gamma \partial_x \phi_k)(x)\phi_k(x)\mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x$$

Integrate by parts $\displaystyle\int_a^b (\gamma(x)\partial_x \phi_k(x))^2 \mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x$

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i, j \leq N$:

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PROOF (SKETCH) Now : $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$?

CONTINOUS CASE : STEP 2

Integrate by parts $\displaystyle\int_a^b (\gamma(x)\partial_x\phi_k(x))^2 \mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x$

Use the expression $\displaystyle\phi_k(x) + \frac{1}{\sqrt{\lambda_k}}\partial_x\phi_k(x) \geq C_1$

$$\int_a^b \lambda_k(\phi_k(x))^2 + (\gamma(x)\partial_x\phi_k(x))^2 \mathrm{d}x \geq \lambda_k C_2$$

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i, j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \quad (2)$$

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PROOF (SKETCH) Now : $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$?

CONTINOUS CASE : STEP 2

Integrate by parts $\boxed{\int_a^b (\gamma(x)\partial_x \phi_k(x))^2 \mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x}$

Use the expression $\phi_k(x) + \frac{1}{\sqrt{\lambda_k}} \partial_x \phi_k(x) \geq C_1$

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Use the expression $\displaystyle\phi_k(x) + \frac{1}{\sqrt{\lambda_k}} \partial_x \phi_k(x) \geq C_1$

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Use the expression $\phi_k(x) + \frac{1}{\sqrt{\lambda_k}}\partial_x\phi_k(x) \geq C_1$

$$\int_\omega (\phi_k(x))^2 \mathrm{d}x \geq \int_a^b (\phi_k(x))^2 \mathrm{d}x \geq C_3$$