Study of the minimal time of null-controllability of Grushin's equation controlled on a vertical strip

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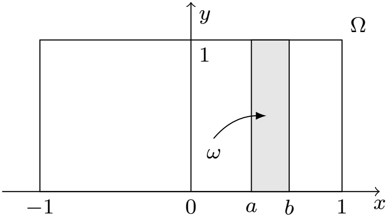
GRUSHIN'S EQUATION

Degenerate elliptic operator : $\boxed{A := -\partial_{xx} - x^2\partial_{yy}}$



$$(G)\begin{cases} \partial_t f + Af = \mathbf{1}_\omega u \text{ on } (0,T)\times\Omega \\ f(0) = f^0 \in L^2(\Omega) \text{ on } \Omega \\ f = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases}$$

- Beauchard, Cannarsa, Guiglielmi, Null controllability of Grushin-type operators in dimension two, 2014.

Minimal time : $T^* \geq \frac{a^2}{2}$.

We want to show that $T^* = \frac{a^2}{2}$.
$\rightarrow$ Moments method.
$\rightarrow$ Study of spectral properties of the operator in 1D : $A_n := -\partial_{xx} + n^2\pi^2x^2$.

FORWARD EQUATION

$$(F.Eq)\begin{cases} \partial_t g - \partial_{xx} g - x^2 \partial_{yy} g = 0, & (t,x,y) \in (0,\infty)\times\Omega \\ g(t,x,y) = 0, & (t,x,y)\in(0,\infty)\times\partial\Omega \\ g(0,x,y) = g^0(x,y), & (x,y)\in\Omega, \end{cases}$$

OBSERVABILITY INEQUALITY

$\exists C>0$, $\forall g^0 \in L^2(\Omega)$,

$$\int_\Omega g^2(T,x,y)dxdy \le C^2 \int_0^T \int_\omega g^2(t,x,y)dxdydt, \qquad \text{(OBS)}$$

Decompose g in Fourier series : $g(t,x,y) = \sum_{n\in\mathbb{N}^*} g_n(t,x)\phi_n(y)$,
with $\phi_n(y) = \sqrt{2}\sin(n\pi y)$ and $g_n(t,x) = \int_0^1 g(t,x,y)\phi_n(y)dy$.
IF :

$$\boxed{\exists c>0,\ \forall g^0\in L^2,\ \forall n\in\mathbb{N}^*,\ \int_{-1}^1 g_n^2(T,x)dx \le c^2\int_0^T\int_a^b g_n^2(t,x)dxdt} \qquad \text{(OBS.n)}$$

then (OBS) holds.

$$\exists c > 0,\ \forall g^0 \in L^2,\ \forall n \in \mathbb{N}^*,\ \int_{-1}^{1} g_n^2(T,x)dx \leq c^2 \int_0^T \int_a^b g_n^2(t,x)dxdt \qquad \text{(OBS.n)}$$

Translate (OBS.n) in terms of control

- Beauchard, Cannarsa, Guiglielmi, Null controllability of Grushin-type operators in dimension two, 2014.

$$\begin{cases} \partial_t g_n - \partial_{xx} g_n + (n\pi)^2 x^2 g_n = u_n(t,x) 1_\omega(x), & (t,x) \in (0,T) \times (-1,1), \\ g_n(t,\pm 1) = 0, & t \in (0,T), \\ g_n(0,x) = \displaystyle\int_0^1 g_0(x,y)\phi_n(y)dy, & x \in (-1,1). \end{cases} \qquad (1)$$

IF :

$$\exists c > 0,\ \forall n \in \mathbb{N}^*,\ \|u_n\|_{L^2((0,T)\times\omega)} \leq c$$

then (OBS.n) holds.
→ Infinite number of control problems !

Definition: Biorthogonal family

Let $\Sigma := (\sigma_k)_{k\geq 1}$ be a sequence of positive real numbers.
Biorthogonal family for Σ, $(q_j^\Sigma)_{j\geq 1}$:

$$\forall k, j \geq 1,\ q_j^\Sigma \in L^2(0,T), \quad \int_0^T e^{-\sigma_k(T-t)} q_j^\Sigma(t)\mathrm{d}t = \delta_{k,j}.$$

Let $n \in \mathbb{N}^*$, denote by $\Lambda_n := (\lambda_{k,n})_{k\geq 1}$ the spectrum of $A_n := -\partial_{xx} + n^2\pi^2x^2$ and $(g_{k,n})_{k\geq 1}$ the corresponding eigenfunctions. Take :

$$u_n(t,x) := \sum_{j\geq 1} \left(1_{(a,b)}(x) g_{j,n}(x)\right) q_j^{\Lambda_n}(t)\alpha_{j,n},$$

with

$$\alpha_{j,n} := -e^{-\lambda_{j,n}T} \times \frac{\displaystyle\int_{-1}^1 g_{n,0}(x) g_{j,n}(x)dx}{\displaystyle\int_a^b g_{j,n}^2(x)dx}$$

Then u_n solves the null-control problem (1).

1. Existence of $(q_j^{\Lambda_n})_{j\geq 1}$?
2. Uniform bounds on $(u_n)_{n\geq 1}$?

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2. $\|u_n\|^2_{L^2((0,T)\times\Omega)} \leq \|g^0\|^2_{L^2(\Omega)} \sum_{j\geq 1} e^{-2\lambda_{j,n}T} \|q_j^{\Lambda_n}\|^2_{L^2(0,T)} / \|g_{j,n}\|^2_{L^2(a,b)}$

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Existence and bound on the biorthogonal family

Definition : set of sequences $\mathcal{L}(\rho, \mathcal{N})$

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.
Denote by $\mathcal{L}(\rho, \mathcal{N})$ the set of sequences $\Sigma = (\sigma_k)_{k \geq 1}$ such that :

- $\forall k \geq 1,\ \sigma_{k+1} - \sigma_k \geq \rho,$
- $\forall \varepsilon > 0,\ \displaystyle\sum_{k=\mathcal{N}(\varepsilon)}^{\infty} \frac{1}{\sigma_k} \leq \varepsilon.$

Theorem [Fattorini-Russel, 1974]

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.

$$\forall \varepsilon > 0,\ \exists K_\varepsilon > 0,\ \boxed{\forall \Sigma \in \mathcal{L}(\rho, \mathcal{N})},\ \exists (q_k^\Sigma)_{k \geq 1},\ \forall k \geq 1,\ \|q_k^\Sigma\|_{L^2} \leq K_\varepsilon \exp(\varepsilon \sigma_k).$$

Recap : What we need to do

1. Lower bound for : $\|g_{j,n}\|^2_{L^2(a,b)}$.
2. UNIFORM (!) gap condition :

$$\boxed{\exists c > 0,\, \forall n \in \mathbb{N}^*,\, \forall k \in \mathbb{N}^*, \lambda_{k+1,n} - \lambda_{k,n} \geq c}$$

.

3. Existence of $\mathcal{N}$ such that : $\forall \varepsilon > 0$, $\forall n \in \mathbb{N}^*$, $\sum_{k=\mathcal{N}(\varepsilon)}^{\infty} \frac{1}{\lambda_{k,n}} \leq \varepsilon$.

2. and 3. $\Rightarrow \forall n \in \mathbb{N}^*$, $\Lambda_n \in \mathcal{L}(\rho, \mathcal{N}) \Rightarrow$ existence + uniform bound on $(q_j^{\Lambda_n})_{j\geq 1}$.

Just use Proposition 4.5 of :

- Beauchard, Miller, Morancey, 2D Grushin-type equations : minimal time and null controllable data. J. Differential equations, 259(11): 5813-5845, 2015.

Proposition

Let $L > 1$ and $\delta \in (0, a)$ such that $a + 3\delta < 1$. There exists $C > 0$ such that for all $\tilde{\delta} > 0$, $q \in C^1([-1,1], \mathbb{R}^+) \setminus \{0\}$, $(w_0, w_1) \in H_0^1 \times L^2(-1,1)$, we have :

$$\|w_0\|^2_{H_0^1} + \|w_1\|^2_{L^2} \leq C(1 + \|\tilde{q}\|_{L^\infty(-1,1)}) O_W(a, q, \tilde{\delta}, \delta) \int_{-L}^{L} \int_{\omega_{a,1}} (w_s^2 + w^2)(s,x)\,dx\,ds, \tag{2}$$

where :

1. $\tilde{q} : x \in (-1,1) \mapsto q(x) + \tilde{\delta}^2 \|q\|_{L^\infty(-1,1)}$,
2. $O_W(a, q, \tilde{\delta}, \delta) = \max\left(e^{\int_0^{a+2\delta} [M(y) + 2\sqrt{\tilde{q}(y)}]dy}, e^{\int_{-a-2\delta}^{0} [M(y) + 2\sqrt{\tilde{q}(y)}]dy} \right)$ *with* $M : x \in (-1,1) \mapsto \frac{|\tilde{q}'(x)|}{\tilde{q}(x)}$, $\omega_{a,1} = (-1,-a) \cup (a,1)$,
3. *and finally w is the solution of the wave equation :*

$$\begin{cases} w_{ss} - w_{xx} + q(x)w = 0, & (s,x) \in (-L,L) \times (-1,1), \\ w(s,\pm 1) = 0, & s \in (-L,L), \\ (w, w_s)(0,x) = (w_0, w_1)(x), & x \in (-1,1). \end{cases} \tag{3}$$

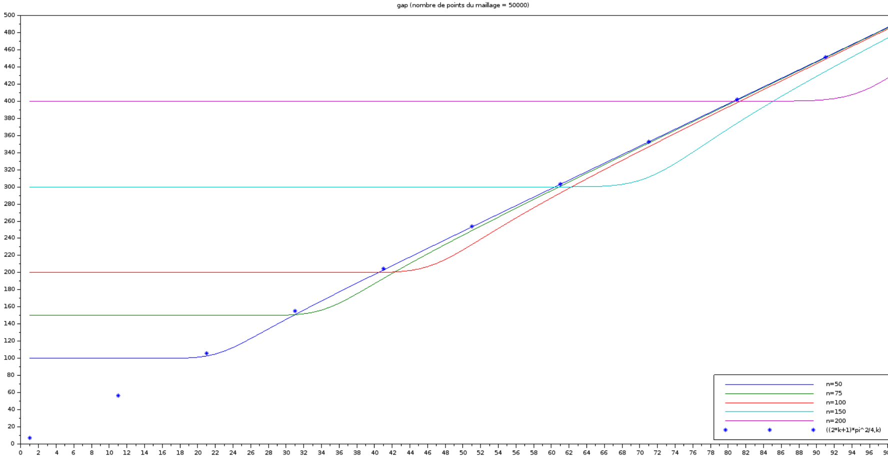
Take $q = (n\pi)^2 x^2$, $w_0 = 0$ and $w_1 = \sqrt{\lambda_{k,n}} g_{k,n}$. Then we have :

$$w(s,x) = \sin(\sqrt{\lambda_{k,n}} s) g_{k,n}(x).$$

and we get :

$$\boxed{\frac{Ce^{-n\pi(a+2\delta)^2}}{n^2\pi^2} \leq \int_a^1 g_{k,n}^2(x)dx} \tag{2}$$

2. Existence of uniform gap : $\lambda_{k+1,n} - \lambda_{k,n} \geq c$?



We define $\mu_k := 2k - 1$.

A general inequality

$$\forall k, n \geq 1, \quad \lambda_{k,n} \geq n\pi(2k-1)$$

Low frequencies

Let $\tau \in (0, \frac{\pi}{2})$. There exist $n_0 \in \mathbb{N}^*$, $c_1 > 0$ and $c_2 > 0$, three constants which only depend on τ, such that for all $n \geq n_0$, $k \leq \tau n$,

$$\mu_k \leq \frac{\lambda_{k,n}}{n\pi} \leq \mu_k + c_1 n^{5/4} e^{-c2n},$$

Hence :

$$\boxed{\lambda_{k+1,n} - \lambda_{k,n} \geq 2n\pi - c_1 n^{5/4} e^{-c2n}}$$

(Small improvement : the same bound holds for $k \leq \frac{\pi}{2} n - cn^{1/3+\varepsilon}$.)

High frequencies

When $k \geq \frac{\pi}{2} n + 1$,

$$\boxed{\lambda_{k+1,n} - \lambda_{k,n} \geq \pi}$$

Intermediate frequencies

When $\frac{\pi}{2} n - cn^{1/3+\varepsilon} \leq k \leq \frac{\pi}{2} n$: ???

3. Existence of function $\mathcal{N}$

Can be established easily.

THANK YOU FOR YOUR ATTENTION