Spectral analysis of an elliptic operator and application in control theory.

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Outline.

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THE EQUATION UNDER CONSIDERATION

Discrete control theory on a semi-discretized parabolic equation on $\Omega = (0,1)$.

$\mathcal{A}^h$: discretization of $\mathcal{A} = -\partial_x \gamma \partial_x \cdot + q\cdot$

$$\begin{cases} \bullet \quad \gamma \in C^2(\Omega), \gamma \geq \gamma_{min} > 0, \\ \bullet \quad q \in C^0(\Omega). \end{cases}$$

$$\begin{cases} \partial_t y^h(t) + \mathcal{A}^h y^h(t) = V_{\mathrm{d}}^h(t) \mathbf{1}_\omega, \text{ on } (0,T), \ (\omega \subset \Omega), \\ y^h(0) = y^{h,0} \in \mathbb{R}^N, \\ y_0^h(t) = 0, \text{ on } (0,T), \\ y_{N+1}^h(t) = V_{\mathrm{b}}^h(t), \text{ on } (0,T), \end{cases}$$

Find $V_{\mathrm{d}}^h \in L^2(0,T;\mathbb{R}^N)$ OR $V_{\mathrm{b}}^h \in L^2(0,T;\mathbb{R})$:

- $y^h(T) = 0$
- V_{d}^h, V_{b}^h uniformly bounded w.r.t. h.

CONTINUOUS PROBLEM

$$\begin{cases} \partial_t y(t,x) + \mathcal{A}y(t,x) = 1_\omega(x)V_d(t,x), \text{ in } (0,T)\times\Omega \\ y(t,x) = 0 \text{ in } (0,T)\times\{0,1\} \\ y(0,x) = y^0(x) \in L^2(\Omega) \text{ in } \Omega. \end{cases}$$

THE MOMENTS METHOD

- $\begin{cases} \Lambda := (\lambda_k)_{k\geq 1}, \\ (\phi_k)_{k\geq 1} \end{cases} \rightarrow$ eigenelements of $\mathcal{A}$.
- $y(T) = 0 \rightarrow$ Moments problem in $L^2((0,T)\times\omega)$:

$$\boxed{-\left\langle y^0, e^{-\lambda_k T}\phi_k \right\rangle_{H^{-1}\times H_0^1} = \int_0^T \int_\omega V_d(t,x) e^{-\lambda_k (T-t)} \phi_k(x) \mathrm{d}x\mathrm{d}t}, \forall k \geq 1$$

- $(q_l^\Lambda)_{l\geq 1}$ biorthogonal family of $(e^{-\lambda_k(T-t)})_{k\geq 1}$ i.e. :

$$\int_0^T e^{-\lambda_k(T-t)} q_l^\Lambda(t)\mathrm{d}t = \delta_{l,k}, \ \forall l,k \geq 1.$$

- We set :

$$V_d(t,x) = \sum_{k\geq 1} \boxed{\alpha_k} q_k^\Lambda(t)\phi_k(x)$$

- Reinjecting :

$$V_d(t,x) = \sum_{k\geq 1} \left(- \frac{\left\langle y^0, \phi_k \right\rangle_{H^{-1}\times H_0^1} e^{-\lambda_k T}}{\|\phi_k\|^2_{L^2(\omega)}} \right) q_k^\Lambda(t)\phi_k(x)$$

CONTINUOUS PROBLEM

$$\begin{cases} \partial_t y(t,x) + \mathcal{A} y(t,x) = 1_\omega(x) V_d(t,x), \text{ in } (0,T)\times\Omega \\ y(t,0) = 0 \text{ in } (0,T) \\ y(t,1) = V_b(t) \text{ in } (0,T) \\ y(0,x) = y^0(x) \in H^{-1}(\Omega) \text{ in } \Omega. \end{cases}$$

THE MOMENTS METHOD

$$V_d(t,x) = \sum_{k\geq 1} \left(- \frac{\langle y^0, \phi_k \rangle_{H^{-1}\times H_0^1} e^{-\lambda_k T}}{\|\phi_k\|^2_{L^2(\omega)}} \right) q_k^\Lambda(t) \phi_k(x)$$

$$V_b(t) = \sum_{k\geq 1} \left(- \frac{\langle y^0, \phi_k \rangle_{H^{-1}\times H_0^1} e^{-\lambda_k T}}{\gamma(1)\partial_x \phi_k(1)} \right) q_k^\Lambda(t) \phi_k(x)$$

REMAINING QUESTIONS

- existence of $(q_k^\Lambda)_{k\geq 1}$?
- convergence of the series : $\begin{cases} \bullet \quad \|\phi_k\|^2_{L^2(\omega)} \geq \dots \\ \bullet \quad |\partial_x \phi_k(1)| \geq \dots \\ \bullet \quad \|q_k^\Lambda\|_{L^2(0,T)} \leq \dots \end{cases}$

About existence and bounds on $(q_k^\Lambda)_{k\geq 1}$

Definition : set of sequences $\mathcal{L}(\rho,\mathcal{N})$

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.
Denote by $\mathcal{L}(\rho,\mathcal{N})$ the set of sequences $\Sigma = (\sigma_k)_{k\geq 1}$ such that :

- $\forall k \geq 1,\ \sigma_{k+1} - \sigma_k \geq \rho,$
- $\forall \varepsilon > 0,\ \displaystyle\sum_{k=\mathcal{N}(\varepsilon)}^{\infty} \frac{1}{\sigma_k} \leq \varepsilon.$

Theorem [Fattorini-Russel, 1974]

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.

$$\forall \varepsilon > 0,\ \exists K_\varepsilon > 0,\ \boxed{\forall \Sigma \in \mathcal{L}(\rho,\mathcal{N})},\ \exists (q_k^\Sigma)_{k\geq 1},\ \forall k \geq 1,\ \|q_k^\Sigma\|_{L^2} \leq K_\varepsilon \exp(\varepsilon\sigma_k).$$

where (q_k^Σ) is a biorthogonal family for Σ.

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REMAINING QUESTIONS

1. $\forall \varepsilon > 0$, $\|\phi_k\|^2_{L^2(\omega)} \geq e^{-\lambda_k \varepsilon}$?
2. $\forall \varepsilon > 0$, $|\partial_x \phi_k(1)| \geq e^{-\lambda_k \varepsilon}$?

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3. $\exists \rho > 0, \mathcal{N}, \quad \Lambda \in \mathcal{L}(\rho,\mathcal{N})$?

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EXAMPLE $\gamma = 1$, $q = 0$

$\lambda_k = \pi^2 k^2$, $\phi_k(x) = \sqrt{2}\sin(k\pi x)$.

1. $\|\phi_k\|^2_{L^2(a,b)} \to b - a$
2. $|\partial_x \phi_k(1)| \geq Ck$
3. $\lambda_{k+1} - \lambda_k \geq Ck$ and $\sum_{k\geq 1} \frac{1}{\lambda_k} < \infty$.

How to estimate the eigenelements

REMAINING QUESTIONS

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2. $\forall \varepsilon > 0,\ |\partial_x \phi_k(1)| \geq e^{-\lambda_k \varepsilon}$?
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Poschel - Trubowitz. Inverse Spectral Theory.

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3. $\exists \rho > 0, \mathcal{N}, \quad \Lambda \in \mathcal{L}(\rho, \mathcal{N})$?

ALTERNATIVE PROOF TRANSPOSABLE TO THE DISCRETE SETTING.

Lemma 1

Let $f : \Omega \to \mathbb{R}$ be a continuous function and $\lambda > 0$. Suppose $u : \Omega \to \mathbb{R}$ satisfies

$$\mathcal{A}u(x) = \lambda u(x) + f, \forall x \in \Omega.$$

Then the following equation holds

$$\boxed{U'(x) = M(x)U(x) + Q(x)U(x) + F(x)},$$

where

$$U(x) = \begin{pmatrix} u(x) \\ \sqrt{\frac{\gamma(x)}{\lambda}}u'(x) \end{pmatrix} \text{ and } F(x) = \begin{pmatrix} 0 \\ -\frac{f(x)}{\sqrt{\gamma(x)\lambda}} \end{pmatrix}$$

and

$$M(x) = \begin{pmatrix} 0 & \sqrt{\frac{\lambda}{\gamma(x)}} \\ -\sqrt{\frac{\lambda}{\gamma(x)}} & 0 \end{pmatrix} \text{ and } Q(x) \text{ is uniformly bounded.}$$

REMAINING QUESTIONS

1. $\forall \varepsilon > 0,\ \|\phi_k\|^2_{L^2(\omega)} \geq e^{-\lambda_k \varepsilon}$?
2. $\forall \varepsilon > 0,\ |\partial_x \phi_k(1)| \geq e^{-\lambda_k \varepsilon}$?
3. $\exists \rho > 0, \mathcal{N}, \quad \Lambda \in \mathcal{L}(\rho, \mathcal{N})$?

ALTERNATIVE PROOF TRANSPOSABLE TO THE DISCRETE SETTING.

Lemma 2

$\exists C_1(q,\gamma) > 0, C_2(q,\gamma,\omega) > 0$ s.t. $\forall k \geq 1$,

$$\frac{1}{\lambda_k}|\partial_x \phi_x(1)| \geq C_1 \mathcal{R}_k \text{ and } \|\phi_k\|^2_{L^2(\omega)} \geq C_2 \mathcal{R}_k,$$

where $\mathcal{R}_k = \inf_{x,y\in\Omega} \frac{|\phi_k(x)|^2 + \frac{\gamma(x)}{\lambda_k}|\phi_k'(x)|^2}{|\phi_k(y)|^2 + \frac{\gamma(y)}{\lambda_k}|\phi_k'(y)|^2}$

•To answer questions 1. and 2., use Lemma 1 with $(u = \phi_k, f = 0)$ and Lemma 2.
•To answer question 3., use Lemma 1 with $u(x) = \phi_k'(1)\phi_{k+1}(x) - \phi_{k+1}'(1)\phi_k(x)$ and $f(x) = \phi_{k+1}'(1)\phi_k(x)(\lambda_{k+1} - \lambda_k)$.

How to estimate the eigenelements

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1. $\forall \varepsilon > 0,\ \|\phi_k\|^2_{L^2(\omega)} \geq e^{-\lambda_k \varepsilon}$?
2. $\forall \varepsilon > 0,\ |\partial_x \phi_k(1)| \geq e^{-\lambda_k \varepsilon}$?
3. $\exists \rho > 0, \mathcal{N}, \quad \Lambda \in \mathcal{L}(\rho, \mathcal{N})$?

ALTERNATIVE PROOF TRANSPOSABLE TO THE DISCRETE SETTING.

Theorem 1

$\exists C_1(q,\gamma) > 0, C_2(q,\gamma,\omega) > 0$ s.t. $\forall k \geq 1$,

1. $\|\phi_k\|^2_{L^2(\omega)} \geq C_2$
2. $|\phi_k'(1)| \geq C_1 k$
3. $\lambda_{k+1} - \lambda_k \geq C_1 k.$

What about the discrete setting ?

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Discrete control theory on a semi-discretized parabolic equation on $\Omega = (0,1)$.

$\boxed{\mathcal{A}^h \,:\, \text{discretization of } \mathcal{A} = -\partial_x \gamma \partial_x \cdot + q\cdot}$, with $\gamma \in C^2(\Omega), q \in C^0(\Omega)$.

$$\begin{cases} \partial_t y^h(t) + \mathcal{A}^h y^h(t) = V_{\mathrm{d}}^h(t)\mathbf{1}_\omega, \text{ on } (0,T), \ (\omega \subset \Omega), \\ y^h(0) = y^{h,0} \in \mathbb{R}^N, \\ y_0^h(t) = 0, \text{ on } (0,T), \\ y_{N+1}^h(t) = V_{\mathrm{b}}^h(t), \text{ on } (0,T), \end{cases}$$

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WHAT WAS DONE BEFORE

1998, López and Zuazua

- semi-discretized heat equation : $\mathcal{A}^h = -\Delta^h$,
- uniform mesh,
- boundary null-control problem : V_{b}^h,
- in space dimension 1 : $\Omega = (0,1)$.

THE EQUATION UNDER CONSIDERATION

Discrete control theory on a semi-discretized parabolic equation on $\Omega = (0,1)$.

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WHAT WAS DONE BEFORE

2010, Boyer, Hubert and Le Rousseau

- semi-discretized parabolic equation : $\mathcal{A}^h = (-\partial_x(\gamma\partial_x \cdot))^h$,
- distributed control problem : V_{d}^h, ($\phi(h)$-null control)
- in space dimension ≥ 1,
- discrete Carleman estimates.

WHAT WE DO

Extend their work to :

- Cascade system of parabolic equations: $\begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix}$ with $\begin{pmatrix} \text{control} \\ 0 \end{pmatrix}$
- boundary and distributed controls : V_{d}^h, V_{b}^h,
- BUT : in space dimension 1.

Outline.

DISCRETE PROBLEM

$$(\mathrm{P}^h) \begin{cases} (y^h)'(t) + \mathcal{A}^h y^h(t) = 0, \text{ on } (0,T), \\ y^h(0) = y^{h,0} \in \mathbb{R}^N \\ y^h_0(t) = 0, \text{ on } (0,T), \\ y^h_{N+1}(t) = 0, \text{ on } (0,T). \end{cases}$$

ELLIPTIC OPERATOR

- $\mathcal{A}^h := \left(-\frac{\partial}{\partial x}\left(\gamma \frac{\partial}{\partial x}\cdot\right) + q\right)^h$,
- $(\mathcal{A}^h y^h)_j = -\frac{1}{h}\left(\gamma_{j+1/2}\frac{y^h_{j+1} - y^h_j}{h} - \gamma_{j-1/2}\frac{y^h_j - y^h_{j-1}}{h}\right) + q_j y^h_j$
- Denote by $(\Lambda^h := (\lambda^h_k)_{k=1}^N, (\phi^h_k)_{k=1}^N)$ the eigenelements of $\mathcal{A}^h$, $\|\phi^h_k\|_h = 1$.
- Quasi-uniform mesh : $\Theta_h := \frac{\max_{i \in [\![0,N]\!]} h_{i+1/2}}{\min_{i \in [\![0,N]\!]} h_{i+1/2}}$ is bounded.

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$$(\mathrm{P}^h)\begin{cases}(y^h)'(t)+\mathcal{A}^h y^h(t)=0+\gamma_{N+1/2}\dfrac{V_b^h(t)}{h^2}\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}, \text{ on } (0,T),\\ y^h(0)=y^{h,0}\in\mathbb{R}^N\\ y_0^h(t)=0, \text{ on } (0,T),\\ y_{N+1}^h(t)=V_{\mathrm{b}}^h(t)\in L^2(0,T;\mathbb{R}), \text{ on } (0,T).\end{cases}$$

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PROPERTY OF THE SOLUTION

- $$\int_0^T \left(e^{-\lambda_k^h (T-t)} \phi_k^h, \left[(y^h)'(t) + \mathcal{A}^h y^h(t) = 0 + \gamma_{N+1/2} \frac{V_b^h(t)}{h^2} \mathbf{e_N} \right] \right) \mathrm{d}t,$$
- Integrate by parts,

$$\left(y^h(T), \phi_k^h \right) - \left(y_0^h, e^{-\lambda_k^h T} \phi_k^h \right) = -\gamma_{N+1/2} \left(\frac{0 - (\phi_k^h)_N}{h} \right) \int_0^T e^{-\lambda_k^h (T-t)} V_b^h(t) \mathrm{d}t$$

$$y^h(T) = 0$$

$$\Updownarrow$$

$$\forall k \in \{1, \dots N\}, \; - \left(y_0^h, e^{-\lambda_k^h T} \phi_k^h \right) = -\gamma_{N+1/2} \left(\frac{0 - (\phi_k^h)_N}{h} \right) \int_0^T e^{-\lambda_k^h (T-t)} V_b^h(t) \mathrm{d}t$$

MOMENT PROBLEM

Find V_d^h and V_b^h, uniformly bounded in h, such that:

$$\forall k \in \{1, \dots N\}, \; - \left(y_0^h, e^{-\lambda_k^h T} \phi_k^h \right) = \begin{cases} -\gamma_{N+1/2} \dfrac{0 - (\phi_k^h)_N}{h} \displaystyle\int_0^T e^{-\lambda_k^h (T-t)} \overbrace{V_b^h(t)}^{\in \mathbb{R}} \mathrm{d}t \\ \displaystyle\int_0^T e^{-\lambda_k^h (T-t)} (\underbrace{V_d^h(t)}_{\in \mathbb{R}^N}, \mathbf{1}_\omega \phi_k^h) \mathrm{d}t \end{cases}$$

$$-\left(y_0^h, e^{-\lambda_k^h T}\phi_k^h\right) = \begin{cases} \displaystyle\int_0^T e^{-\lambda_k^h (T-t)} \left(V_{\mathrm{d}}^h(t), \mathbf{1}_\omega \phi_k^h\right) \mathrm{d}t \\ \displaystyle -\gamma_{N+1/2} \left(\frac{0-(\phi_k^h)_N}{h}\right) \int_0^T e^{-\lambda_k^h (T-t)} V_{\mathrm{b}}^h(t) \mathrm{d}t \end{cases}$$

POSSIBLE EXPRESSIONS FOR THE CONTROLS

$$V_{\mathrm{d}}^h(t) = \sum_{j=1}^N \frac{-\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)}{\|\mathbf{1}_\omega \phi_j^h\|_h^2} \phi_j^h q_j^{\Lambda^h}(t),$$

$$V_{\mathrm{b}}^h(t) = \sum_{j=1}^N \frac{\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)}{\gamma_{N+1/2}\left(\frac{0-\phi_{j,N}^h}{h}\right)} q_j^{\Lambda^h}(t).$$

IT REMAINS TO PROVE

- Uniform bounds on V_{b}^h and V_{d}^h $\Leftarrow$ $\boxed{\|\mathbf{1}_\omega \phi_j^h\|_h^2 \geq \text{?} \text{ and } \left(\frac{0-\phi_{j,N}^h}{h}\right) \geq \text{?}}$
- Bounds on $(q_j^{\Lambda^h})_{j\geq 1}$ for all $h > 0$

POSSIBLE EXPRESSIONS FOR THE CONTROLS

$$V_{\mathrm{d}}^h(t) = \sum_{j=1}^{N} \frac{-\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)}{\|\mathbf{1}_\omega \phi_j^h\|_h^2} \phi_j^h q_j^{\Lambda^h}(t),$$

$$V_{\mathrm{b}}^h(t) = \sum_{j=1}^{N} \frac{\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)}{\gamma_{N+1/2}\left(\frac{0-\phi_{j,N}^h}{h}\right)} q_j^{\Lambda^h}(t).$$

IT REMAINS TO PROVE

- Uniform bounds on V_{b}^h and V_{d}^h $\Leftarrow$ $\boxed{\|\mathbf{1}_\omega \phi_j^h\|_h^2 \geq ? \text{ and } \left(\frac{0 - \phi_{j,N}^h}{h}\right) \geq ?}$
- Bounds on $(q_j^{\Lambda^h})_{j\geq 1}$ for all $h > 0$ $\Leftarrow$ $\boxed{\text{find } \rho, \mathcal{N} : \forall h > 0, \Lambda^h \in \mathcal{L}(\rho, \mathcal{N})}$

Theorem [Fattorini-Russel, 1974]

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.

$$\forall \varepsilon > 0, \exists K_\varepsilon > 0, \boxed{\forall \Sigma \in \mathcal{L}(\rho, \mathcal{N})}, \exists (q_k^\Sigma)_{k\geq 1}, \forall k \geq 1, \|q_k^\Sigma\|_{L^2} \leq K_\varepsilon \exp(\varepsilon \sigma_k).$$

where (q_k^Σ) is a biorthogonal family for Σ.

WHEN $\gamma = 1$ AND $q = 0$: $\mathcal{A} = -\Delta$ (UNIFORM MESH)

Theorem [López-Zuazua,1998], boundary control problem.

The moment method on the semi-discretized heat equation gives uniformly bounded boundary controls :

$$\|V_{\mathrm{b}}^h\|_{L^2(0,T;\mathbb{R})} \leq C_T \|y^{h,0}\|.$$

for the null-control problem (P^h).

PROOF Explicit expression for the eigenelements !

$$\begin{cases} \phi_k^h = (\sin(j\pi hk))_{j=1}^N, \text{ we can estimate } \left| \frac{0-(\phi_k^h)_N}{h} \right| \geq \frac{2}{\pi}\sqrt{\lambda_k^h} \\ \forall k \in \{1,\dots,N\},\ \lambda_k^h = \frac{4}{h^2}\sin^2\left(\frac{\pi hk}{2}\right) \end{cases}$$

Extend the sequence:

$$(\lambda_k^h)_{k\geq 1} = \begin{cases} \dfrac{4}{h^2}\sin^2(\dfrac{\pi hk}{2}), \text{ for } k \in \{1,\dots N\}, & \text{(discrete eigenvalues)} \\ k^2\pi^2, \text{ for } k \geq N+1. & \text{(continuous eigenvalues)} \end{cases}$$

There exist $\rho > 0$, and $\mathcal{N}$ such that

$$\boxed{\forall h > 0,}\ \Lambda^h := (\lambda_k^h)_{k\geq 1} \in \mathcal{L}(\rho, \mathcal{N}).$$

WHEN $\gamma = 1$ AND $q = 0$: $\mathcal{A} = -\Delta$ (UNIFORM MESH)

Theorem [López-Zuazua,1998], boundary control problem.

The moment method on the semi-discretized heat equation gives uniformly bounded boundary controls :

$$\|V_{\mathrm{b}}^h\|_{L^2(0,T;\mathbb{R})} \leq C_T \|y^{h,0}\|.$$

for the null-control problem (P^h).

γ AND q IN THE GENERAL CASE ?

Can one obtain the same results with a general operator $\mathcal{A} = -\frac{\partial}{\partial x}\left(\gamma \frac{\partial}{\partial x} \cdot\right) + q \cdot$?
No explicit formulae for the eigenelements.

STRATEGY

- Find ρ and $\mathcal{N}$ such that : $\forall h > 0$, $\Lambda^h \in \mathcal{L}(\rho, \mathcal{N})$.
- Find lower bounds on $\left|\frac{0-(\phi_k^h)_N}{h}\right|$ and $\|\mathbf{1}_\omega \phi_k^h\|$.

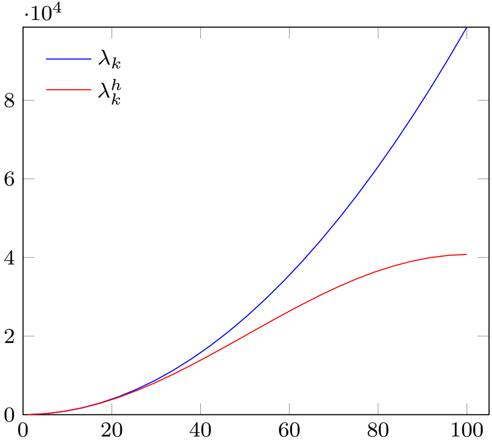
Outline.

PROBLEM

Find sharp lower-bounds for : $\left|\frac{0-(\phi_k^h)_N}{h}\right|$ and $\|\mathbf{1}_\omega \phi_k^h\|$.

Basic approach: try to use numerical analysis $\lambda_k^h \approx \lambda_k$.

$\lambda_k^h \approx \lambda_k \implies$ Gap property only for a portion of the spectrum.



Discrete bounds on eigenvectors.

PROBLEM

Find sharp lower-bounds for : $\left|\frac{0-(\phi_k^h)_N}{h}\right|$ and $\|\mathbf{1}_\omega \phi_k^h\|$.

Basic approach: try to use numerical analysis $\lambda_k^h \approx \lambda_k$.

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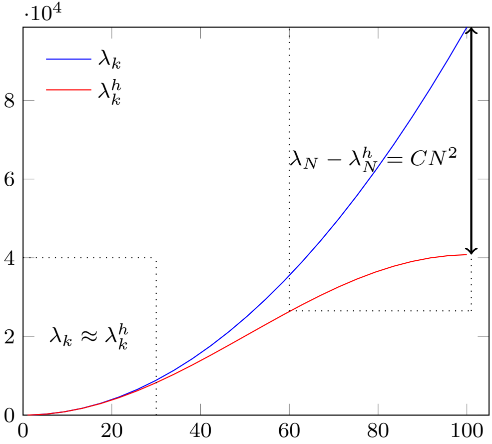


PROBLEM

Find sharp lower-bounds for : $\left|\frac{0-(\phi_k^h)_N}{h}\right|$ and $\|\mathbf{1}_\omega \phi_k^h\|$.

Lemma 2 - in the discrete setting

Let $\beta > 0$ and suppose that $\Theta_h < \beta$.
$\exists C_1(q,\gamma,\beta) > 0, C_2(q,\gamma,\omega,\beta) > 0$ s.t. $\forall k \geq 1$,

$$\frac{1}{\lambda_k^h}\left|\frac{0-(\phi_k^h)_N}{h_{N+1}}\right| \geq C_1 \mathcal{R}_k^h \text{ and } \|\phi_k^h\|^2_{L^2(\omega^h)} \geq C_2 \mathcal{R}_k^h,$$

where $\mathcal{R}_k^h = \min_{i,j\in[\![1,N+1]\!]} \dfrac{\left|\phi_{i,k}^h\right|^2 + \frac{\gamma_{i-1/2}}{\lambda_k^h}\left|\frac{\phi_{i,k}^h - \phi_{i-1,k}^h}{h_{i-1/2}}\right|^2}{|\phi_{j,k}^h|^2 + \frac{\gamma_{j-1/2}}{\lambda_k^h}\left|\frac{\phi_{j,k}^h - \phi_{j-1,k}^h}{h_{j-1/2}}\right|^2}$.

Goal : Find a sharp lower bound of $\mathcal{R}_k$.

PROOF Find a sharp lower bound of $\mathcal{R}_k$.

DISCRETE SETTING

EINGENVALUE PROBLEM FOR $\mathcal{A}^h$

- "ODE" of order 2 : $\mathcal{A}^h u^h = \lambda u^h + f^h \longrightarrow$ system of "ODEs" of order 1.
- **CHANGE OF VARIABLE** $(U^h)_j = \begin{pmatrix} (u^h)_j \\ \frac{(u^h)_j - (u^h)_{j-1}}{h} \frac{\sqrt{\gamma_{j-1/2}}}{\sqrt{\lambda}} \end{pmatrix}$
- Duhamel's formula + Gronwall's lemma : $\forall 1 \leq i,j \leq N$

$$\|(U^h)_j\| \leq \max_{1\leq i,j\leq N} \|S^\lambda_{i\leftarrow j}\| \left(\|(U^h)_i\| + h \sum_{p\in [\![i,j]\!]\setminus\{i\}} \|F^h_p\| \right) \times \exp\left(\max_{1\leq i,j\leq N} \|S^\lambda_{i\leftarrow j}\| h \sum_{p\in [\![i,j]\!]\setminus\{i\}} \|Q^h_p\| \right)$$

where : $S^\lambda_{i,j} = (I_h + hM_{\lambda,i-1})(I_h + hM_{\lambda,i-2})\dots(I_h + hM_{\lambda,j})$,

and : $M_{\lambda,j} := \begin{pmatrix} -h\frac{\lambda}{\gamma_{j+1/2}} & \sqrt{\frac{\lambda}{\gamma_{j+1/2}}} \\ -\sqrt{\frac{\lambda}{\gamma_{j+1/2}}} & 0 \end{pmatrix}$

ESTIMATES ON EIGENVECTORS (questions 1. & 2.)

Take $u^h = \phi_k^h$, $\lambda = \lambda_k^h \to F^h = 0$.

We get,

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \leq \max_{1\leq i,j\leq N} \|S_{i\leftarrow j}^{\lambda}\| \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)}$$

Proposition : Estimates on $S^{\lambda}_{i\leftarrow j}$

Estimates on the semi-group $S^{\lambda}_{i\leftarrow j}$ for all i, j:

- **For any k :** $\|S^{\lambda}_{i\leftarrow j}\| \leq e^{C\sqrt{\lambda}}$,

Proposition : Estimates on the eigenvectors

- **For any k :** $\left|\frac{(\phi^h_k)_N}{h}\right| \geq C_1 e^{-C_2\sqrt{\lambda^h_k}}$ and $h\sum_{jh\in\omega}|(\phi^h_k)_j|^2 \geq C_1 e^{-C_2\sqrt{\lambda^h_k}}$

Proposition : Gap property

- **For any k :** NO UNIFORM GAP PROPERTY.

Discrete bounds on eigenvectors.

Proposition : Estimates on $S^\lambda_{i\leftarrow j}$

Estimates on the semi-group $S^\lambda_{i\leftarrow j}$ for all i, j:

- **For any k :** $\|S^\lambda_{i\leftarrow j}\| \leq e^{C\sqrt{\lambda}}$,

Define

$$k^h_{max,\varepsilon} := \max\left\{k \in \{1, \dots N\};\ \lambda^h_k < \frac{4}{h^2}\gamma_{min}(1-\varepsilon)\right\}.$$

- **For $k \leq k^h_{max,\varepsilon}$:** $\|S^\lambda_{i\leftarrow j}\| \leq \frac{1}{\delta_\varepsilon}$

Proposition : Estimates on the eigenvectors

- **For any k :** $\left|\frac{(\phi^h_k)_N}{h}\right| \geq C_1 e^{-C_2\sqrt{\lambda^h_k}}$ and $h\sum_{jh\in\omega}|(\phi^h_k)_j|^2 \geq C_1 e^{-C_2\sqrt{\lambda^h_k}}$
- **For $k \leq k^h_{max,\varepsilon}$:** $\left|\frac{(\phi^h_k)_N}{h}\right| \geq \delta_\varepsilon\sqrt{\lambda^h_k}$ and $h\sum_{jh\in\omega}|(\phi^h_k)_j|^2 \geq \delta_\varepsilon$

Proposition : Gap property

- **For any k :** NO UNIFORM GAP PROPERTY.
- **For $k \leq k^h_{max,\varepsilon}$:** $\lambda^h_{k+1} - \lambda^h_k \geq \delta_\varepsilon$

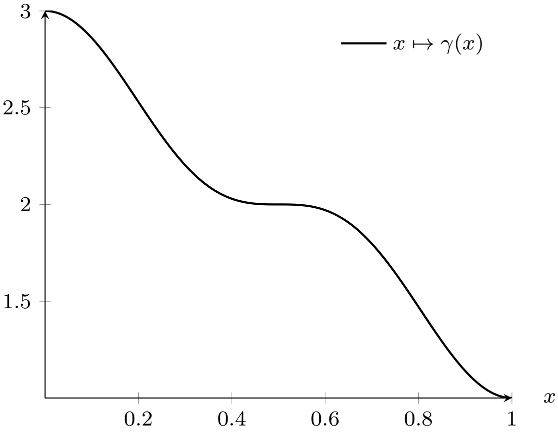
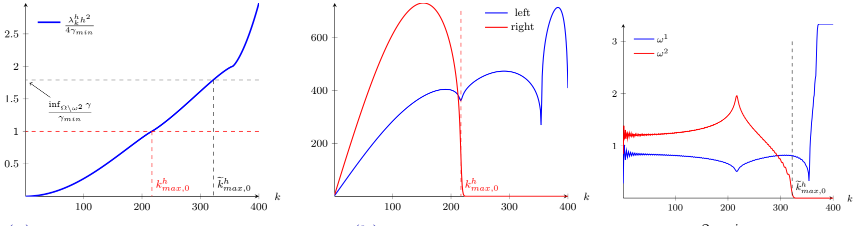


Figure: Case 1 - $\gamma(x) = 2 + \cos(\pi x)^3$.

Numerical simulations : $\gamma(x) = 2 + \cos(\pi x)^3$, $q = 0$.



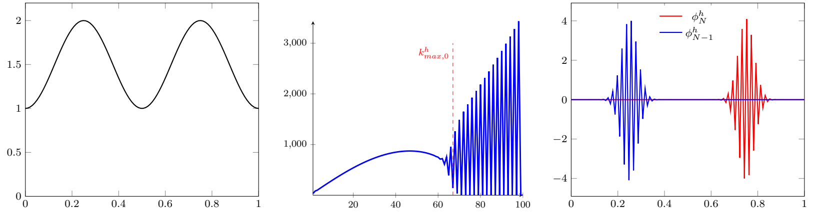
(a) Normalized discrete spectrum (b) Normal derivatives (c) $L^2(\omega_h^i)$ norms

Figure: Case 1 - N = 400

N	$k^h_{max,\varepsilon}$	$I^h_l(\cdot)$		$I^h_r(\cdot)$		$I^h_1(\cdot)$		$I^h_2(\cdot)$		$\Delta^h(\cdot)$	
		$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N
50	26	2.99	2.99	7.07	1.01_{-23}	0.29	0.29	0.86	8.71_{-29}	56.82	56.82
100	52	2.99	2.99	7.08	2.46_{-51}	0.28	0.28	0.85	1.26_{-59}	56.89	56.89
200	104	2.99	2.99	7.08	4.16_{-107}	0.28	0.28	0.85	1.97_{-121}	56.91	56.91
300	156	2.99	2.99	7.08	4.22_{-163}	0.28	0.28	0.84	2.70_{-183}	56.91	56.91
400	208	2.99	2.99	7.08	3.47_{-219}	0.28	0.28	0.84	3.50_{-245}	56.91	56.91

Table: Case 1 - behavior as $h \to 0$

Numerical simulations : $\gamma = 2 - \cos(2\pi x)^2, q = 0.$



(a) The diffusion coefficient γ (b) $k \mapsto |\lambda^h_{k+1} - \lambda^h_k|$ (c) The last two eigenfunctions

Figure: Case 2 - N = 100

N	$k^h_{max,\varepsilon}$	$I^h_l(\cdot)$		$I^h_r(\cdot)$		$I^h_1(\cdot)$		$I^h_2(\cdot)$		$\Delta^h(\cdot)$	
		$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N
50	32	6.39	1.74_{-3}	6.39	1.74_{-3}	0.56	0.56	0.6	0.6	33.53	4.51_{-8}
100	64	6.41	6.85_{-15}	6.41	7.18_{-30}	0.59	1.75_{-30}	0.59	2.83_{-42}	33.58	2.91_{-11}
200	126	6.42	3.02_{-63}	6.42	3.80_{-14}	0.58	8.90_{-87}	0.58	2.81_{-30}	33.59	2.91_{-11}
300	187	6.42	8.41_{-15}	6.42	9.47_{-97}	0.61	4.41_{-30}	0.58	1.60_{-131}	33.59	1.16_{-10}
400	250	6.42	2.50_{-130}	6.42	5.30_{-15}	0.58	7.38_{-176}	0.58	6.02_{-30}	33.59	6.98_{-10}

Table: Case 2 - behavior as $h \to 0$

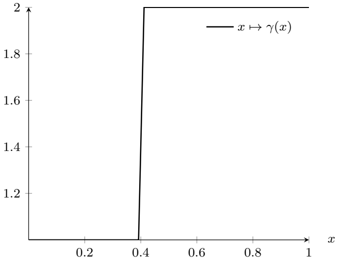


Figure: Case 3 - $\gamma(x) = 1_{]0,0.4[} + 2 \times 1_{]0.4,1[}$, $q = 0$.

N	$k^h_{max,\varepsilon}$	$I^h_l(\cdot)$		$I^h_r(\cdot)$		$I^h_1(\cdot)$		$I^h_2(\cdot)$		$\Delta^h(\cdot)$	
		$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N	$k^h_{max,\varepsilon}$	N
50	35	4.41	1.89_{-14}	3.79	3.79	0.55	2.43_{-11}	0.12	0.12	44.89	44.89
100	68	5.37	9.21_{-30}	3.79	3.79	0.68	7.18_{-20}	5.87_{-2}	5.87_{-2}	44.62	44.62
200	131	5.37	2.19_{-60}	3.79	3.79	0.67	4.53_{-36}	8.86_{-2}	2.79_{-2}	44.47	44.47
300	194	5.37	5.22_{-91}	3.79	3.79	0.67	6.63_{-52}	9.81_{-2}	1.88_{-2}	44.42	44.42
400	257	5.37	1.25_{-121}	3.79	3.79	0.67	1.37_{-67}	0.1	1.42_{-2}	44.4	44.4

Table: Case 3 - behavior as $h \to 0$

Outline.

EXPRESSIONS OF THE CONTROLS

$$V_{\mathrm{d}}^h(t) = \sum_{j=1}^{N} \frac{-\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)_h}{\|\mathbf{1}_\omega \phi_j^h\|_h^2} \phi_j^h q_j^{\Lambda^h}(t),$$

$$V_{\mathrm{b}}^h(t) = \sum_{j=1}^{N} \frac{\left(y^{h,0}, e^{-\lambda_j^h T}\phi_j^h\right)_h}{\gamma_{N+1/2}\left(\frac{0-\phi_{j,N}^h}{h}\right)} q_j^{\Lambda^h}(t).$$

RECALL THE STRATEGY

- Find lower bounds on $\left|\frac{0-(\phi_k^h)_N}{h}\right|$ or $\|\mathbf{1}_\omega \phi_k^h\|$: **OK.**
- Find ρ and $\mathcal{N}$ such that : $\forall h > 0,\ \Lambda^h \in \mathcal{L}(\rho, \mathcal{N})$: KO.

TO SUM UP

- **For all k,** $\|\mathbf{1}_\omega \phi_j^h\|_h^2 \geq C_1 e^{-C_2\sqrt{\lambda_k^h}}$.
- **For all k,** $\left|\frac{(\phi_k^h)_N}{h}\right| \geq C_1 e^{-C_2\sqrt{\lambda_k^h}}$.
- If $k \leq k_{max,\varepsilon}^h$, then $\lambda_{k+1}^h - \lambda_k^h \geq \delta_\varepsilon$.

Partial controlability result.

Theorem [A.-Boyer-Morancey, 2016]

We say that system (P^h) is $\underline{\phi(h)\text{-null controllable}}$ if :
$\forall T > 0$, there exists a control V_{d}^h (or V_{b}^h) satisfying

$$\forall h > 0, \ \|V_{\mathrm{d}}^h\| \leq C\|y^{h,0}\| \quad (\text{or } \|V_{\mathrm{b}}^h\| \leq C\|y^{h,0}\|)$$

and such that the corresponding solution verifies:

$$\forall h > 0, \ \|y^h(T)\|^2 \leq \phi(h)\|y^{h,0}\|^2.$$

Let any function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$$\liminf_{h \to 0} [h^2 \log(\phi(h))] > -8\gamma_{min} T,$$

Then, $\underline{\text{on a uniform mesh}}$ system (P^h) is $\phi(h)$-null controllable.

Remarks

The solution satisfies in fact: $\forall h > 0, \ \|y^h(T)\| \leq \|y^{h,0}\| C_1 e^{-\frac{C_2 T}{h^2}}$.

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and such that the corresponding solution verifies:

$$\forall h > 0,\ \|y^h(T)\|^2 \leq \phi(h)\|y^{h,0}\|^2.$$

Let any function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$$\liminf_{h\to 0}[h^{\frac{2}{5}} \log(\phi(h))] > -\alpha T,$$

Then, on a quasi-uniform mesh system (P^h) is $\phi(h)$-null controllable.

Remarks

The solution satisfies in fact: $\forall h > 0,\ \|y^h(T)\| \leq \|y^{h,0}\| C_1 e^{-\frac{C_2 T}{h^2}}$.

System of two parabolic equations in one space dimension, $\Omega = (0, L)$.
Only one control force on the first equation (distributed or boundary).

$$(\mathrm{S}^h) \begin{cases} (y^h)'(t) + \begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix} y^h(t) = \begin{pmatrix} V_{\mathrm{d}}^h \mathbf{1}_\omega \\ 0 \end{pmatrix} + \gamma_{N+1/2} \dfrac{V_b^h(t)}{h^2} \begin{pmatrix} \mathbf{e_N} \\ 0 \end{pmatrix}, \text{ on } (0,T), \\ y^h(0) = y^{h,0} \in (\mathbb{R}^N)^2 \\ y_0^h(t) = 0, \text{ on } (0,T), \end{cases}$$

Note that the second equation is controlled by the solution to the first one.

System of two parabolic equations in one space dimension, $\Omega = (0, L)$.
Only one control force on the first equation (distributed or boundary).

$$(\mathrm{S}^h)\begin{cases} (y^h)'(t) + \begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix} y^h(t) = \begin{pmatrix} V_{\mathrm{d}}^h \mathbf{1}_\omega \\ 0 \end{pmatrix} + \gamma_{N+1/2} \dfrac{V_b^h(t)}{h^2} \begin{pmatrix} \mathbf{e_N} \\ 0 \end{pmatrix}, \text{ on } (0,T), \\ y^h(0) = y^{h,0} \in (\mathbb{R}^N)^2 \\ y_0^h(t) = 0, \text{ on } (0,T), \end{cases}$$

Theorem [A.-Boyer-Morancey, 2016]

Let any function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$$\liminf_{h\to 0} [h^2 \log(\phi(h))] > -8\gamma_{min} T,$$

Then, on a uniform mesh (S^h) system is $\phi(h)$-null controllable.

Remarks

The Carleman technics employed by [2010, Boyer, Hubert and Le Rousseau] cannot be used here.

Controllability of a parabolic system in cascade.

System of two parabolic equations in one space dimension, $\Omega = (0, L)$.
Only one control force on the first equation (distributed or boundary).

$$(\mathrm{S}^h)\begin{cases} (y^h)'(t) + \begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix} y^h(t) = \begin{pmatrix} V_{\mathrm{d}}^h \mathbf{1}_\omega \\ 0 \end{pmatrix} + \gamma_{N+1/2} \dfrac{V_b^h(t)}{h^2} \begin{pmatrix} \mathbf{e_N} \\ 0 \end{pmatrix}, \text{ on } (0, T), \\ y^h(0) = y^{h,0} \in (\mathbb{R}^N)^2 \\ y_0^h(t) = 0, \text{ on } (0, T), \end{cases}$$

Theorem [A.-Boyer-Morancey, 2016]

Let any function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$$\liminf_{h \to 0} [h^{\frac{2}{5}} \log(\phi(h))] > -\alpha T,$$

Then, on a quasi-uniform mesh (S^h) system is $\phi(h)$-null controllable.

Remarks

The Carleman technics employed by [2010, Boyer, Hubert and Le Rousseau] cannot be used here.

System of two parabolic equations in one space dimension, $\Omega = (0, L)$.
Only one control force on the first equation (distributed or boundary).

$$(\mathrm{S}^h) \begin{cases} (y^h)'(t) + \begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix} y^h(t) = \begin{pmatrix} V_{\mathrm{d}}^h \mathbf{1}_\omega \\ 0 \end{pmatrix} + \gamma_{N+1/2} \dfrac{V_b^h(t)}{h^2} \begin{pmatrix} \mathbf{e_N} \\ 0 \end{pmatrix}, \text{ on } (0,T), \\ y^h(0) = y^{h,0} \in (\mathbb{R}^N)^2 \\ y_0^h(t) = 0, \text{ on } (0,T), \end{cases}$$

Elements of proof.

Main difference with the scalar case :

- Operator $\begin{pmatrix} \mathcal{A}^h & 0 \\ 1 & \mathcal{A}^h \end{pmatrix}$ is not diagonalizable $\Rightarrow$ we use the Jordan form.
- Existence + estimates of a biorthogonal family for

$$\left\{ e^{-\lambda_k^h t} \right\}_{k \geq 1} \cup \left\{ t e^{-\lambda_k^h t} \right\}_{k \geq 1}.$$

$\square$

SUM UP

We have built an elementary approach:

- to solve the $\phi(h)$-null controllability control problem for a large class of parabolic equations,
- which applies on quasi-uniform meshes,
- which applies on a parabolic cascade system,
(with fewer controls than equations)
- only valid in 1D.

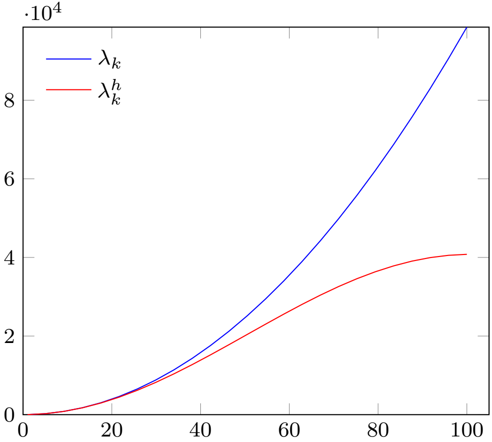
PERSPECTIVES

Cascade systems with variable coefficients.

Thank you for your attention !

Basic approach: try to use numerical analysis $\lambda_k^h \approx \lambda_k$.

$\lambda_k^h \approx \lambda_k \implies$ Gap property only for a portion of the spectrum.



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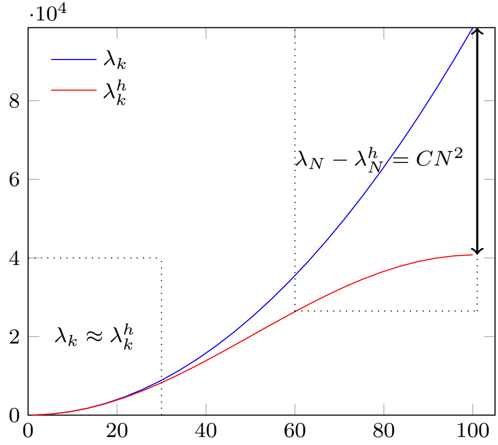

Bonus slide 2 : Extension of [Fattorini-Russel, 1974]

Definition : set of sequences $\mathcal{L}(\rho,\mathcal{N})$

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.
Denote by $\mathcal{L}(\rho,\mathcal{N})$ the set of sequences $\Sigma = (\sigma_k)_{k\geq 1}$ such that :

- $\forall k \geq 1,\ \sigma_{k+1} - \sigma_k \geq \rho,$
- $\forall \varepsilon > 0,\ \displaystyle\sum_{k=\mathcal{N}(\varepsilon)}^{\infty} \frac{1}{\sigma_k} \leq \varepsilon.$

Theorem [Fattorini-Russel, 1974]

Let $\rho > 0$ and $\mathcal{N} : \mathbb{R}^+ \to \mathbb{N}$.

$$\forall \varepsilon > 0,\ \exists K_\varepsilon > 0,\ \boxed{\forall \Sigma \in \mathcal{L}(\rho,\mathcal{N})},\ \exists (q_k^\Sigma)_{k\geq 1},\ \forall k \geq 1,\ \|q_k^\Sigma\|_{L^2} \leq K_\varepsilon \exp(\varepsilon\sigma_k).$$

where (q_k^Σ) is a biorthogonal family for Σ.

[Ammar Khodja - Benabdallah - González Burgos - de Teresa, 2011]
Let $m \in \mathbb{N}$, we have the same results for the family $(t^j e^{-\sigma_k t})_{m\geq j\geq 0, k\geq 1}$.

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i, j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \quad (1)$$

then the following relations holds: $\left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$ and $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$.

PROOF (SKETCH)

$$\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)$$

$$\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left|(\phi_k^h)_j\right| \quad \text{now : } h\sum_{j=1}^{N} \cdot$$

$$\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k. \text{ Take } i = N+1 : \left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$$

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i,j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \quad (1)$$

then the following relations holds: $\left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$ and $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$.

PROOF (SKETCH) Now : $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$?

CONTINOUS CASE : STEP 1

Find a nodal domain (a,b) in ω : $\phi_k(a) = \phi_k(b) = 0$

$$\int_a^b -\partial_x(\gamma \partial_x \phi_k)(x)\phi_k(x)\mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x$$

Integrate by parts $\displaystyle\int_a^b (\gamma(x)\partial_x \phi_k(x))^2 \mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x$

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i,j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \quad (1)$$

then the following relations holds: $\left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$ and $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$.

PROOF (SKETCH) Now : $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$?

CONTINOUS CASE : STEP 2

Integrate by parts $\int_a^b (\gamma(x)\partial_x\phi_k(x))^2 \mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x$

Use the expression $\phi_k(x) + \frac{1}{\sqrt{\lambda_k}}\partial_x\phi_k(x) \geq C_1$

$$\int_a^b \lambda_k(\phi_k(x))^2 + (\gamma(x)\partial_x\phi_k(x))^2 \mathrm{d}x \geq \lambda_k C_2$$

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i, j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \quad (1)$$

then the following relations holds: $\left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$ and $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$.

PROOF (SKETCH) Now : $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$?

CONTINOUS CASE : STEP 2

Integrate by parts $\boxed{\int_a^b (\gamma(x)\partial_x \phi_k(x))^2 \mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x}$

Use the expression $\phi_k(x) + \frac{1}{\sqrt{\lambda_k}} \partial_x \phi_k(x) \geq C_1$

$$\int_a^b \lambda_k (\phi_k(x))^2 + \boxed{(\gamma(x)\partial_x \phi_k(x))^2} \mathrm{d}x \geq \lambda_k C_2$$

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i, j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \qquad (1)$$

then the following relations holds: $\left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$ and $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$.

PROOF (SKETCH) Now : $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$?

CONTINOUS CASE : STEP 2

Integrate by parts $\int_a^b (\gamma(x)\partial_x \phi_k(x))^2 \mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x$

Use the expression $\phi_k(x) + \frac{1}{\sqrt{\lambda_k}} \partial_x \phi_k(x) \geq C_1$

$$\int_a^b 2\lambda_k (\phi_k(x))^2 \mathrm{d}x \geq \lambda_k C_2$$

Lemma

Assume that one can prove that there exists C_k such that $\forall 1 \leq i, j \leq N$:

$$\boxed{\left|(\phi_k^h)_i\right| + \left|\frac{(\phi_k^h)_i - (\phi_k^h)_{i-1}}{h\sqrt{\lambda_k^h}}\right| \geq C_k \left(\left|(\phi_k^h)_j\right| + \left|\frac{(\phi_k^h)_j - (\phi_k^h)_{j-1}}{h\sqrt{\lambda_k^h}}\right|\right)} \quad (1)$$

then the following relations holds: $\left|\frac{0-(\phi_k^h)_N}{h\sqrt{\lambda_k^h}}\right| \geq C_k$ and $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$.

PROOF (SKETCH) Now : $\|\mathbf{1}_\omega \phi_k^h\| \geq C_k$?

CONTINOUS CASE : STEP 2

Integrate by parts $\int_a^b (\gamma(x)\partial_x \phi_k(x))^2 \mathrm{d}x = \lambda_k \int_a^b (\phi_k(x))^2 \mathrm{d}x$

Use the expression $\phi_k(x) + \frac{1}{\sqrt{\lambda_k}} \partial_x \phi_k(x) \geq C_1$

$$\int_\omega (\phi_k(x))^2 \mathrm{d}x \geq \int_a^b (\phi_k(x))^2 \mathrm{d}x \geq C_3$$